Corrected entropy of high dimensional black holes

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(Dated: October 31, 2009)

Using the corrected expression of Hawking temperature derived from the tunneling formalism beyond semiclassical approximation developed by *Banerjee* and *Majhi*[12], we calculate the corrected entropy of a high dimensional Schwarzschild black hole and a 5-dimensional Gauss-Bonnet (GB) black hole. It is shown that the corrected entropy for this two kinds of black hole are in agreement with the corrected entropy formula (2) that derived from tunneling method for a $(n+1)$-dimensional Friedmann-Robertson-Walker (FRW) universe[19]. This feature strongly suggests deep universality of the corrected entropy formula (2), which may not depend on the dimensions of spacetime and gravity theories. In addition, the leading order correction of corrected entropy formula always appears as the logarithmic of the semiclassical entropy, rather than the logarithmic of the area of black hole horizon, this might imply that the logarithmic of the semiclassical entropy is more appropriate for quantum correction than the logarithmic of the area.

One of the intriguing properties of a black hole is that it carries entropy[1]. Understanding this entropy is an enormous challenge in modern physics. Any developments in this direction might lead to important insights into the structure of quantum gravity which includes in particular the notion of "holography" and the emerging notion of "quantum spacetime". There are many approaches to calculate the entropy of a black hole. With the semiclassical approximation, the black hole entropy obeys the celebrated Bekenstein-Hawking area law. When full quantum effect is raised, the area law should undergo corrections, and these corrections can be obtained from field theory methods[2], quantum geometry techniques[3], general statistical mechanical arguments[4], Cardy formula[5], etc[6]. All these approaches show that the corrected entropy formula takes the form

$$S_c = S + \alpha \ln S + \cdots , \qquad (1)$$

where α is a dimensionless constant and S denotes the uncorrected semiclassical entropy of black hole. It is well known that the corrected entropy formula of eq.(1) is universal. This implies that eq.(1) could be valid for all black holes.

Hawking radiation from the horizon of a black hole[7] also provides an approach to calculate thermodynamic entities like temperature and entropy of a black hole. Many different derivations of Hawking radiation exist in the literature. Among these a simple and physically intuitive picture is provided by the tunneling mechanism. It has two variants namely null geodesic method[8] and Hamilton-Jacobi method[9]. Recently, the connection between the anomaly approach and tunneling mechanism

is discussed[10] and the Hawking black body spectrum is obtained from tunneling machanism[11]. However, most of these derivations are confined to the semiclassical approximation. Recently, a general formalism of tunneling beyond semiclassical approximation has been developed by *Banerjee* and *Majhi*[12]. This formalism has been used to investigate the quantum corrections to the semiclassical entropy for various black holes[13, 14, 15, 16, 17]. More interestingly, the corrected entropy formulas of black holes calculated from this formalism all take the form same as eq.(1).

In our recent work[18, 19], this formalism has been extended from black holes to Friedmann-Robertson-Walker (FRW) universe. We have shown that the corrected entropy of apparent horizon for a FRW universe takes the form[19]

$$S_c = S + \alpha_1 \ln S + \sum_{i=2} \frac{\alpha_i}{S^{i-1}} + \texttt{const.} \qquad (2)$$

It is obvious that the first and the second terms have the same form as eq.(1). As pointed out in ref.[19], this corrected entropy formula has three important features:

- Eq.(2) not only holds in Einstein gravity, but also is valid for Gauss-Bonnet gravity, Lovelock gravity, $f(R)$ gravity and scalar-tensor gravity. This feature might imply that the corrected entropy formula of eq.(2) is independent of gravity theories.

- Eq.(2) is derived from the tunneling method in an arbitrary dimensions $(n + 1)$-dimensional FRW spacetime. This might imply that it is independent of dimensions of the spacetime.

- The corrected entropy formulas of different black holes calculated from the tunneling method all take the form same as eq.(2), such as those of the BTZ black hole, Kerr-Newmann black hole, etc.

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These features strongly suggest deep universality of the corrected entropy formula of eq.(2).

Up to now, the given examples of black holes are all confined to low dimensional spacetimes. Since the FRW universe is different from black holes and eq.(2) is derived for a FRW universe, one may ask that whether the corrected entropy formula of eq.(2) is still valid for high dimensional black holes in Einstein gravity or in other gravity theories. In order to answer this question, we investigate the corrected entropy formula of high dimensional black holes in tunneling perspective. We explicitly compute the corrected expressions for the temperature and the entropy for an $(n+1)$-dimensional Schwarzschild black hole and a 5-dimensional Gauss-Bonnet black hole. It is shown that the corrected entropy formula of eq.(2) is also valid for this two kinds of black hole, and therefore the universality of eq.(2) is more plausible.

Consider an $(n + 1)$-dimensional static, spherically symmetric spacetime of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_{n-1}^2, \tag{3}$$

where $d\Omega_{n-1}^2$ denotes the line element of an $(n-1)$-dimensional unit sphere. The horizon of the black hole $r = r_H$ is given by $f(r_H) = g(r_H) = 0$. In this spacetime, a massless scalar particle obeys the Klein-Gordon equation

$$-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)\phi = 0. \tag{4}$$

Note that, in the tunneling framework, the tunneling particle is considered as a spherical shell. For this the trajectory of the tunneling process is radial and therefore only the (r, t) sector of the metric (3) is important. In this case, tunneling of a particle from a black hole can be considered as a two-dimensional quantum process in (r, t) plane. For a two dimensional theory, the standard WKB ansatz for the wave function ϕ can be expressed as

$$\phi(r,t) = \exp\left[\frac{i}{\hbar}I(r,t)\right], \tag{5}$$

where $I(r, t)$ is one particle action which will be expanded in powers of $\hbar$ as

$$I(r,t) = I_0(r,t) + \sum_i \hbar^i I_i(r,t). \tag{6}$$

Here $I_0(r, t)$ is the semiclassical action and the other terms are treat as quantum corrections. Since only the (r, t) sector is relevant and the other dimensions can not affect the tunneling process, the treatment and the result for low dimensional black holes are same as these here. As shown in ref.[12, 14], $I_i(r, t)$ are proportional to $I_0(r, t)$, thus we have

$$I(r,t) = \left(1 + \sum_i \gamma_i \hbar^i\right) I_0(r,t). \tag{7}$$

With this expression of action, the corrected Hawking temperature can be expressed as[12, 14]

$$T_c = T_H \left(1 + \sum_i \gamma_i \hbar^i\right)^{-1}, \tag{8}$$

where

$$T_H = \frac{\hbar}{4}\left(\text{Im}\int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\right)^{-1} \tag{9}$$

is the standard semiclassical Hawking temperature of the black hole.

With the corrected Hawking temperature (8) we now proceed with the calculation of the corrected entropy of black holes. We first consider an $(n + 1)$-dimensional Schwarzschild black hole whose metric has the form[20]

$$\begin{aligned}ds^2 &= -\left(1 - \frac{m}{r^{n-2}}\right)dt^2 + \left(1 - \frac{m}{r^{n-2}}\right)^{-1}dr^2 \\ &+ r^2 d\Omega_{n-1}^2.\end{aligned} \tag{10}$$

The ADM mass of the black hole is given by $M = \frac{(n-1)\Omega_{n-1}m}{16\pi}$. Its semiclassical Hawking temperature can be obtained from eq.(9) as

$$T_H = \frac{(n-2)\hbar}{4\pi r_H}. \tag{11}$$

The mass is related to the horizon radius as

$$M = \frac{(n-1)\Omega_{n-1}}{16\pi}r_H^{n-2}, \tag{12}$$

where $r_H = m^{1/(n-2)}$ is the location of the horizon. In the semiclassical approximation, the entropy of the horizon obeys the Bekenstein-Hawking area law

$$S_{\text{BH}} = \frac{A}{4\hbar}, \tag{13}$$

where $A = \Omega_{n-1}r_H^{n-1}$ is the area of the horizon. With semiclassical temperature (11) and entropy (13), the first law of thermodynamics holds on the horizon

$$T_H dS_{\text{BH}} = dM. \tag{14}$$

From this expression the Bekenstein-Hawking entropy can be computed as

$$S_{\text{BH}} = \int \frac{dM}{T_H}. \tag{15}$$

In the Hawking temperature expression (8), there are un-determined coefficients γ_i. Obviously, γ_i should have the dimension $\hbar^{-i}$. Now, we will perform the following dimensional analysis to express these γ_i in terms of dimensionless constants by invoking some basic macroscopic parameters of high dimensional black hole. In the $(n + 1)$-dimensional spacetime, one sets the units

as $G_{n+1} = c = k_B = 1$, where G_{n+1} is the $(n + 1)$-dimensional gravitation constant. In this setting, the Planck constant $\hbar$ is of the order of l_p^{n-1}, where l_p is the Planck length. Therefore, according to the dimensional analysis, the proportionality constants γ_i have the dimension of $l_p^{i(1-n)}$[14, 17]. For Schwarzschild black hole, the only macroscopic parameter is the radius of horizon r_H. Therefore, one can express the proportionality constants γ_i in terms of black hole parameters as

$$\gamma_i = \alpha_i r_H^{-i(n-1)}, \tag{16}$$

where α_i are dimensionless constants. Now the corrected Hawking temperature can be written as

$$
\begin{aligned}
T_c &= T_H \left(1 + \sum_i \frac{\alpha_i \hbar^i}{(r_H^{n-1})^i} \right)^{-1} \\
&= T_H \left(1 + \sum_i \frac{\tilde{\alpha}_i}{(S_{\text{BH}})^i} \right)^{-1},
\end{aligned} \tag{17}
$$

where $\tilde{\alpha}_i = (\frac{\Omega_{n-1}}{4})^i \alpha_i$ are also dimensionless constants. Note that for Schwarzschild black hole S_{BH} is proportional to the area of horizon, thus it only dependent on the radius of horizon r_H. Things will be a bit different for Gauss-Bonnet black hole while the entropy is not proportional to the area of horizon.

Replace the semiclassical Hawking temperature T_H with the corrected Hawking temperature (17), one can determine the corrected entropy by the integral

$$S_c = \int \frac{dM}{T_c} = \int \frac{dM}{T_H} \left(1 + \sum_i \frac{\tilde{\alpha}_i}{S_{\text{BH}}^i} \right). \tag{18}$$

Integrating the above expression, we obtain the corrected Bekenstein-Hawking entropy of an $(n + 1)$-dimensional Schwarzschild black hole as

$$S_c = S_{\text{BH}} + \tilde{\alpha}_1 \ln S_{\text{BH}} + \sum_{i=2} \frac{\tilde{\alpha}_i}{1-i} \frac{1}{S_{\text{BH}}^{i-1}} + \text{const.} \tag{19}$$

Interestingly the leading order correction is logarithmic in S_{BH}, which is consistent with eqs.(1) and (2).

Now we turn to the Gauss-Bonnet black hole, which is the black hole solution in Gauss-Bonnet gravity. The $(4 + 1)$ dimensional static, spherically symmetric black hole solution in this theory is of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3, \tag{20}$$

where the metric function is

$$f(r) = 1 + \frac{r^2}{2\alpha} \left[1 - \left(1 + \frac{4\alpha m}{r^4} \right)^{1/2} \right]. \tag{21}$$

Here m is related to the ADM mass M by the relationship $M = \frac{3\Omega_3}{16\pi} m$. The event horizon is located at r_H which satisfies

$$r_H^2 + \alpha - m = 0. \tag{22}$$

For the horizon to exist at all, one must have $r_H^2 + 2\alpha > 0$. Thus the ADM mass can be expressed in term of r_H as

$$M = \frac{3\Omega_3}{16\pi} (r_H^2 + \alpha). \tag{23}$$

Substituting the metric (20) into eq.(9), we obtain the Hawking temperature for this black hole

$$T_H = \frac{\hbar}{2\pi} \frac{r_H}{r_H^2 + 2\alpha}. \tag{24}$$

For black holes in Einstein gravity, the entropy of the horizon is proportional to its area. Gauss-Bonnet gravity is the natural generalization of Einstein gravity by including higher derivative correction term, i.e., the Gauss-Bonnet term to the original Einstein-Hilbert action. In this gravity theory, the semiclassical Bekenstein-Hawking entropy-area relationship that the entropy of the horizon is proportional to its area, does not hold anymore. The relationship is now[21]

$$S_{\text{GB}} = \frac{A}{4\hbar} \left(1 + \frac{6\alpha}{r_H^2} \right). \tag{25}$$

But the first law of thermodynamics still holds on the horizon of a Gauss-Bonnet black hole

$$T_H dS_{\text{GB}} = dM. \tag{26}$$

For Schwarzschild black hole, one can express the corrected temperature (8) in terms of S_{BH} as (17). This is always correct because S_{BH} is only dependent on r_H, which is the only independent macroscopic parameter of the Schwarzschild black hole. Unlike Schwarzschild black hole that has only one independent macroscopic parameter r_H, the Gauss-Bonnet black hole have two independent parameters r_H and α. Thus eq.(16) in which the undetermined coefficients γ_i have been expressed in terms of r_H might be not appropriate here. In this case, the most general form of the proportionality constants γ_i can be expressed in terms of r_H and α as

$$\gamma_i = (c_i r_H^3 + d_i r_H \alpha + e_i \alpha^{3/2})^{-i}, \tag{27}$$

where un-determined coefficients c_i, d_i, and e_i are dimensionless constants. Thus the corrected temperature now for Gauss-Bonnet black hole is

$$T_c = T_H \left(1 + \sum_i \frac{\alpha_i \hbar^i}{(c_i r_H^3 + d_i r_H \alpha + e_i \alpha^{3/2})^i} \right)^{-i}. \tag{28}$$

In order to determine c_i, d_i, and e_i, one should treat the independent parameter α as a variable. Due to the difference in α, the semi-classical first law of thermodynamics of Gauss-Bonnet black hole should be modified as

$$T_H dS = dM + \frac{3\Omega_3}{16\pi} \frac{3r_H^2 - 2\alpha}{r_H^2 + 2\alpha} d\alpha, \tag{29}$$

where the added term is just the "work term" induced by the differentiation of α. Now with the corrected temperature (28), the first law of thermodynamics is

$$dS_c = \frac{3\Omega_3}{4\hbar} \frac{T_H}{T_c} \left[(r_H^2 + 2\alpha)dr_H + 2r_H d\alpha \right]. \qquad (30)$$

From the principle of the ordinary first law of thermodynamics one interprets entropy as a state function. In refs.[13, 14, 19], this property of entropy has been used to investigate the first law of thermodynamics and entropy for black holes. The entropy must be a state function means that dS_c has to be an exact differential. As a result the following integrability condition must hold:

$$\frac{\partial}{\partial \alpha} \left[\frac{T_H}{T_c}(r_H^2 + 2\alpha) \right]\bigg|_{r_H} = \frac{\partial}{\partial r_H} \left[2r_H \frac{T_H}{T_c} \right]\bigg|_{\alpha}. \quad (31)$$

From this condition one can easily determine c_i, d_i, and e_i as

$$d_i = 6c_i, \quad e_i = 0. \qquad (32)$$

Substituting above results into (28), it is easy to show that the corrected temperature for Gauss-Bonnet black hole can be expressed as

$$T_c = T_H \left(1 + \sum_i \frac{\tilde{\alpha}_i}{(S_{\text{GB}})^i} \right)^{-1}. \qquad (33)$$

This expression have two important features. First, it involves both the parameters r_H and α to express the un-determined coefficients γ_i. Second, it ensures that the corresponding corrected entropy is a state function when we treat the parameter α as a variable.

With the corrected Hawking temperature (33), the corrected expression of the entropy for this black hole can be determined as

$$S_c = S_{\text{GB}} + \tilde{\alpha}_1 \ln S_{\text{GB}} + \sum_{i=2} \frac{\tilde{\alpha}_i}{1-i} \frac{1}{S_{\text{GB}}^{i-1}} + \texttt{const.} \quad (34)$$

It is clear that this corrected entropy formula is consistent with eq.(2).

Thus we have derived the corrected entropy formula for a high dimensional Schwarzschild black hole and a 5 dimensional Gauss-Bonnet black hole. By using the corrected expression of Hawking temperature derived from tunneling formalism beyond semiclassical approximation and applying the semiclassical first law of thermodynamics for this two black holes, the corrected expressions of the entropy of the horizon are determined. All the high order quantum corrections to the entropy are computed. It is shown that these corrected expressions of entropy are both in agreement with eq.(2). It seems that, the corrected entropy formula of black holes in tunneling perspective does not depend on the dimensions of spacetime and gravity theories. This supports the universality of the corrected entropy formula (2) by tunneling method.

There is another significant point for the corrected entropy, is that it involves the term of the logarithmic of S as the leading order correction. From the Table I, it is easy to see that for a large number of black holes, the leading order corrections in the corrected entropy appears as the logarithmic of the semiclassical entropy S. This feature agrees with the universal quantum corrected entropy expression (1). In Einstein gravity, the corrected entropy expression is usually written as another form

$$S_c = \frac{A}{4\hbar} + \alpha \ln A + \cdots. \qquad (35)$$

Of course, because the entropy of the horizon is proportional to its area in Einstein gravity, this form is consistent with eq.(1). But for other non-Einstein gravity theories, the semiclassical Bekenstein-Hawking entropy-area relationship that the entropy of the horizon is proportional to its area, does not hold anymore. Then, the expression (35) is not valid in these cases. So one can conclude that, in the expression of quantum corrected entropy of the horizon, the logarithmic of the semiclassical entropy is more appropriate than the logarithmic in the area of horizon.

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	Does ln S exist?
3D BTZ black hole[13]	Yes
4D Schwarzchild black hole[12, 14]	Yes
4D Schwarzschild-AdS black hole[12]	Yes
4D Schwarzschild-AdS black hole in Rainrow gravity[15]	Yes
4D Reissner-Nordstrom black hole[14]	Yes
4D Kerr black hole[14]	Yes
4D Kerr-Newmann black hole[14]	Yes
5D Gauss-Bonnet black hole	Yes
$(n+1)D$ Schwarzchild black hole	Yes
$(n+1)D$ FRW universe for various gravity theories[19]	Yes

TABLE I: The leading order correction in the corrected entropy appear as the logarithmic of the semiclassical entropy S for various black holes.

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